FP3 June 2018 (MA)

Q1a) $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

$\dfrac{\sinh x}{\cosh x} = \tanh x = \dfrac{\frac{e^{x} - e^{-x}}{2}}{\frac{e^{x} + e^{-x}}{2}} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$ //

$\times e^{x}$

$= \dfrac{e^{2x} - 1}{e^{2x} + 1}$

b) let $y = \text{artanh}\,\theta$,

$\theta = \tanh(y)$

$\theta = \dfrac{e^{2y} - 1}{e^{2y} + 1}$

$\theta e^{2y} + \theta = e^{2y} - 1$

$\theta e^{2y} - e^{2y} + \theta + 1 = 0$

$e^{2y}(\theta - 1) = -1 - \theta$

$e^{2y} = \dfrac{-1 - \theta}{\theta - 1}$ // $= \dfrac{1 + \theta}{1 - \theta}$ //

$\therefore 2y = \ln\left(\frac{1+\theta}{1-\theta}\right)$

$y = \frac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right)$ //

hence $\operatorname{artanh}\theta = \frac{1}{2}\ln\left(\frac{1+\theta}{1-\theta}\right)$

(Q2a) $y = 5\cosh x - 6\sinh x = 0$

$5\cosh x = 6\sinh x$

$\div \cosh x$: $5 = 6\tanh x$

$\tanh x = \frac{5}{6} = \cancel{e^{2x}}$

$x = \operatorname{artanh}\left(\frac{5}{6}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{5}{6}}{1-\frac{5}{6}}\right)$

$= \frac{1}{2}\ln\left(\frac{11/6}{1/6}\right) = \frac{1}{2}\ln 11$

$= \boxed{\ln\sqrt{11}}$

Q3a) $M - \lambda I = \begin{pmatrix} 3-\lambda & k & 2 \\ -1 & -\lambda & 1 \\ 1 & k & 1-\lambda \end{pmatrix}$

$$\det(M-\lambda I) = 3-\lambda \begin{vmatrix} -\lambda & 1 \\ k & 1-\lambda \end{vmatrix} - k \begin{vmatrix} -1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & -\lambda \\ 1 & k \end{vmatrix} = 0$$

$$= (3-\lambda)\left[-\lambda + \lambda^2 - k\right] - k\left[\lambda - 1 - 1\right] + 2\left[\lambda - k\right] = 0$$

$$= -3\lambda + 3\lambda^2 - 3k + \lambda^2 - \lambda^3 + \lambda k - k\lambda + \cancel{2k} + 2\lambda - \cancel{2k}$$

$$= -\lambda^3 + 4\lambda^2 + \lambda(-3 + k - k + 2) - 3k$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 3k = 0$$

$$\Rightarrow \quad \lambda^3 - 4\lambda^2 + \lambda + 3k = 0$$

$\lambda - 3$ is a factor,

$$\begin{array}{r|l} & \lambda^2 - \lambda - 2 \\ \hline \lambda - 3 & \lambda^3 - 4\lambda^2 + \lambda + 3k \\ & \lambda^3 - 3\lambda^2 \\ \hline & 0 \quad -\lambda^2 + \lambda \\ & \quad -\lambda^2 + 3\lambda \\ \hline & \quad 0 \quad -2\lambda + 3k \\ & \quad\quad -2\lambda + 6 \\ \hline & \quad\quad 0 \quad 0 \end{array}$$

remainder should be 0

$3k - 6 = 0$

$k = 2$

b) $\Rightarrow (\lambda-3)(\lambda^2-\lambda-2) = 0$

$\Rightarrow (\lambda-3)(\lambda-2)(\lambda+1) = 0$

$$\boxed{\begin{array}{l}\lambda = 2\\ \lambda = -1\end{array}}$$

c) $Mx = \lambda x$

$$\begin{pmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$3x + 2y + 2z = 3x$ — ①

$\therefore y + z = 0$ //

$-x + z = 3y$ — ②

$x + 2y + z = 3z$ — ③

①: let $y = 1$, $z = -1$ //

$\hookrightarrow$ ②: $x = z - 3y = -1 - 3(1)$
$= -4$ //

So a corresponding eigenvector is...

$$\boxed{\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}}$$

(Q4a) $y = \operatorname{arsinh} x + x(x^2+1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + (x^2+1)^{\frac{1}{2}} + x\left(\tfrac{1}{2}\right)(x^2+1)^{-\frac{1}{2}}(2x$$

$$= \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1}$$

$$= \frac{1+x^2+x^2+1}{\sqrt{x^2+1}} = \frac{2(x^2+1)}{\sqrt{x^2+1}}$$

$$= \boxed{2\sqrt{x^2+1}}$$

b) length $= \displaystyle\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + 4(x^2+1) = 4x^2+5$$ //

$\therefore$ length $= \displaystyle\int_0^1 \sqrt{4x^2+5}\,dx$

c) $\int_0^1 \sqrt{4x^2+5}\, dx$

$x = \frac{\sqrt{5}}{2}\sinh u$

$\frac{dx}{du} = \frac{\sqrt{5}}{2}\cosh u$

$dx = \frac{\sqrt{5}}{2}\cosh u\, du$

x	u
0	0
1	$\text{arsinh}\left(\frac{2}{\sqrt{5}}\right)$

$\Downarrow$

$$= \int_0^{\text{as}\frac{2}{\sqrt{5}}} \left(\sqrt{4\left(\frac{5}{4}\sinh^2 u\right)+5}\right) \times \frac{\sqrt{5}}{2}\cosh u\, du$$

$$= \frac{\sqrt{5}}{2}\int_0^{\text{as}\frac{2}{\sqrt{5}}} \sqrt{5(1+\sinh^2 u)} \times \cosh u\, du$$

$\cosh^2 u = 1+\sinh^2 u$

$$= \frac{\sqrt{5}}{2}\int_0^{\text{as}\frac{2}{\sqrt{5}}} \sqrt{5} \times \sqrt{\cosh^2 u} \times \cosh u\, du = \frac{5}{2}\int_0^{\text{as}\frac{2}{\sqrt{5}}} (\cosh^2 u)\, du$$

$\cosh 2u = 2\cosh^2 u - 1$

$\frac{\cosh 2u + 1}{2} = \cosh^2 u$ //

$$\Rightarrow \frac{5}{4}\int_0^{\text{as}\left(\frac{2}{\sqrt{5}}\right)} (1+\cosh 2u)\, du = \frac{5}{4}\left[u + \frac{1}{2}\sinh 2u\right]_0^{\text{as}\frac{2}{\sqrt{5}}}$$

$$= \frac{5}{4}\left[\text{as}\frac{2}{\sqrt{5}} + \frac{1}{2}\sinh\left(2\,\text{as}\frac{2}{\sqrt{5}}\right)\right]$$

$$\text{arsinh}\left(\frac{2}{\sqrt{5}}\right) = \ln\left(\frac{2}{\sqrt{5}} + \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + 1}\right)$$

$$= \ln(\sqrt{5}) //$$

$$\Rightarrow \frac{5}{4}\left[\ln(\sqrt{5}) + \frac{1}{2}\sinh(\ln 5)\right]$$

$$\Rightarrow \frac{5}{8} \times 2\ln\sqrt{5} + \frac{1}{2}\left(\frac{12}{5}\right)$$

$$\Rightarrow \frac{5}{8}\ln 5 + \frac{12}{10} = \boxed{\frac{5}{8}\ln 5 + \frac{6}{5}}$$

Q5a) $I_n = \int x^n (x+8)^{1/2}\,dx$

By Parts

$\frac{dv}{dx} = (x+8)^{1/2}$ $\quad v = \frac{2}{3}(x+8)^{3/2}$

$u = x^n$ $\quad u' = nx^{n-1}$

$$\Rightarrow \left[\frac{2}{3}x^n(x+8)^{3/2}\right] - \frac{2}{3}n\int\left[x^{n-1}(x+8)^{3/2}\right]dx$$

$$x^{n-1}(x+8)^{3/2} = x^{n-1}(x+8)(x+8)^{1/2}$$

$$= x^n(x+8)^{1/2} + 8x^{n-1}(x+8)^{1/2}$$

(Q5a) $I_n = \int x^n (x+8)^{1/2} \, dx$

By Parts : $\frac{dv}{dx} = (x+8)^{\frac{1}{2}} \rightarrow v = \frac{2}{3}(x+8)^{3/2}$

$u = x^n \rightarrow nx^{n-1} = u'$

$\Rightarrow I_n = \left[\frac{2}{3}x^n (x+8)^{3/2}\right] - \frac{2}{3}n \int \left[x^{n-1}(x+8)^{3/2}\right] dx$

$\int x^{n-1}(x+8)^{3/2} \, dx = \int x^{n-1}(x+8)\sqrt{x+8} \, dx$

$= \int x^n \sqrt{x+8} \, dx + 8\int x^{n-1}\sqrt{x+8} \, dx$

$\therefore I_n = \frac{2}{3}x^n (x+8)^{3/2} - \frac{2}{3}n \left[\int x^n \sqrt{x+8} \, dx + 8\int x^{n-1}\sqrt{x+8} \, dx\right]$

$I_n = \frac{2}{3}x^n (x+8)^{3/2} - \frac{2}{3}n \left[I_n + 8I_{n-1}\right]$

$3I_n = 2x^n (x+8)^{3/2} - 2n[I_n] - 2n[8I_{n-1}]$

$(3+2n)I_n = 2x^n (x+8)^{3/2} - 16n I_{n-1}$

$I_n = \frac{2x^n (x+8)^{3/2}}{3+2n} - \frac{16n}{3+2n} I_{n-1}$

b) $I_2 = \left[\frac{2x^2(x+8)^{3/2}}{3+4}\right]_0^{10} - \frac{16(2)}{7} I_1$

$= \frac{200}{7}(18)^{3/2} - \frac{32}{7} I_1$

$= \frac{10800}{7}\sqrt{2} - \frac{32}{7} I_1$

$I_1 = \left[\frac{2}{5}x(x+8)^{3/2}\right]_0^{10} - \frac{16}{5} I_0$

$= \frac{20}{5}(10+8)^{3/2} - \frac{16}{5} I_0$

$= 4(54\sqrt{2}) - \frac{16}{5} I_0$

$= 216\sqrt{2} - \frac{16}{5} I_0$

$I_0 = \int_0^{10} (x+8)^{1/2}\,dx = \left[\frac{2}{3}(x+8)^{3/2}\right]_0^{10}$

$= \frac{2}{3}(54\sqrt{2}) - \frac{2}{3}(16\sqrt{2}) = \frac{76}{3}\sqrt{2}$

$$\therefore I_2 = \frac{10800}{7}\sqrt{2} - \frac{32}{7}\left[216\sqrt{2} - \frac{16}{5}\left(\frac{76}{3}\sqrt{2}\right)\right]$$

$$= \frac{10800}{7}\sqrt{2} - \frac{6912}{7}\sqrt{2} + \frac{38912}{105}\sqrt{2}$$

$$= \left[\frac{(10800 - 6912)}{7} + \frac{38912}{105}\right]\sqrt{2}$$

$$= \boxed{\frac{97232}{105}\sqrt{2}}$$

Q6a) $l_2 : \quad r = \underset{\boxed{a}}{\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}} + \mu \underset{\boxed{b}}{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}$

$l_1 : \quad r = \underset{\boxed{c}}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}} + \lambda \underset{\boxed{d}}{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}$

assume intersection:

$$\begin{pmatrix} -1 + \mu \\ 4 + \mu \\ 1 + 3\mu \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ 0 + 3\lambda \\ 2 - \lambda \end{pmatrix} \begin{matrix} \text{— } ① \\ \text{— } ② \\ \text{— } ③ \end{matrix}$$

② : $\mu = 3\lambda - 4$ // $\rightarrow$ ① : $-1 + 3\lambda - 4 = 1 + 2\lambda$

$\therefore \lambda = 6$ //

② : $\mu = 3(6) - 4 = 14$ //

put $\lambda = 6$ into ③ : $\mu = \frac{2-6}{3} - 1 = -\frac{5}{3}$

So values of μ and λ are not consistent so l_1 and l_2 don't intersect.

looking at direction vectors,

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \neq a \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$ where a is an arbitrary constant.

So l_1 and l_2 <u>aren't</u> parallel.

Since l_1 and l_2 don't intersect and are not parallel then we can conclude they are indeed skew.

b) Shortest distance between 2 skew lines

$$= \left| \frac{(a-c)\cdot(b\times d)}{|b\times d|} \right|$$

$$(a-c) = \begin{pmatrix} -1-1 \\ 4-0 \\ 1-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$$

$$(b\times d) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix}$$

$$(a-c)\cdot(b\times d) = \begin{pmatrix}-2\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}-10\\7\\1\end{pmatrix}$$

$$= 20+28-1 = 47$$

$$|b\times d| = \sqrt{10^2+7^2+1^2} = 5\sqrt{6}$$

$$\therefore \text{distance} = \frac{47}{5\sqrt{6}} = \boxed{\frac{47\sqrt{6}}{30}}$$

c) Π contains the points $\underbrace{\begin{pmatrix}3\\8\\13\end{pmatrix}}_{A}$ and $\underbrace{\begin{pmatrix}1\\0\\2\end{pmatrix}}_{B}$ and the line l_1.

we must find a normal to Π.

$$\overrightarrow{AB} = \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}3\\8\\13\end{pmatrix} = \begin{pmatrix}-2\\-8\\-11\end{pmatrix}$$

$\boxed{\overrightarrow{AB}} \times \begin{pmatrix}2\\3\\-1\end{pmatrix}$ will yield a normal to Π

$$\begin{pmatrix}-2\\-8\\-11\end{pmatrix}\times\begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}41\\-24\\10\end{pmatrix}$$

So... $r\cdot\begin{pmatrix}41\\-24\\10\end{pmatrix} = \begin{pmatrix}41\\-24\\10\end{pmatrix}\cdot\begin{pmatrix}3\\8\\13\end{pmatrix} = 41(3)-24(8)+130$

$$\boxed{r \cdot \begin{pmatrix} 41 \\ -24 \\ 10 \end{pmatrix} = 61}$$

So cartesian eqn :

$$\boxed{41x - 24y + 10z = 61}$$

Q7a) $ae = 3$ $\qquad \frac{a}{e} = \frac{25}{3}$

so $a = \frac{25}{3}e$ //

$\therefore \frac{25}{3}e^2 = 3$

$b^2 = a^2(1-e^2)$

$b^2 = 25(1 - \frac{9}{25})$

$b^2 = 16$

$e^2 = \frac{9}{25}$ so $e = \boxed{\frac{3}{5}}$ //

$\therefore a = \frac{25}{3} \times \frac{3}{5} = \boxed{5}$ //

$\therefore \boxed{\frac{x^2}{25} + \frac{y^2}{16} = 1}$ //

b) $\frac{x^2}{25} + \frac{(mx+c)^2}{16} = 1$

$\frac{x^2}{25} + \frac{m^2x^2 + 2mxc + c^2}{16} = 1$

$\times 400$: $16x^2 + 25m^2x^2 + 50mcx + 25c^2 = 400$

$(16 + 25m^2)x^2 + (50mc)x + 25c^2 - 400 = 0$

hence $(25m^2 + 16)x^2 + (50mc)x + 25(c^2 - 16) = 0$

c) l is a tangent to E $\therefore$ $b^2 - 4ac = 0$

$\left.\begin{array}{l} a = 16 + 25m^2 \\ b = 50mc \\ c = 25(c^2 - 16) \end{array}\right\}$ $\begin{array}{l} b^2 - 4ac = 0 \\ (50mc)^2 - 4(16 + 25m^2)(25)(c^2 - 1 \end{array}$

$\Rightarrow 2500m^2c^2 - 100(16c^2 - 256 + 25m^2c^2 - 400m^2) = 0$

$\Rightarrow 2500\cancel{m^2}c^2 - 1600c^2 + 25600 - 2500\cancel{m^2}c^2 + 40000m^2 = 0$

$\Rightarrow -1600c^2 + 25600 + 40000m^2 = 0$

$\div -1600$: $c^2 - 16 - 25m^2 = 0$

$\therefore$ $\boxed{c^2 = 25m^2 + 16}$

d) Area OAB $= \left|\frac{1}{2}(OA)(OB)\right|$

$y = mx + c$ and $c^2 = 25m^2 + 16$

$x = 0$: $y = c$

$y = 0$: $x = -\frac{c}{m}$

$\therefore$ Area $= \left|\frac{1}{2}(c)\left(-\frac{c}{m}\right)\right| = \left|\frac{-c^2}{2m}\right|$

$= \frac{c^2}{2m} = \frac{25m^2+16}{2m}$

e) $A = \frac{25m^2}{2m} + \frac{16}{2m} = \frac{25m}{2} + 8m^{-1}$

$\therefore \frac{dA}{dm} = \frac{25}{2} - 8m^{-2} = 0$

$\frac{25}{2} = \frac{8}{m^2}$

$\frac{2}{25} = \frac{m^2}{8}$

$\therefore m^2 = \frac{16}{25} \rightarrow m = \frac{4}{5}$ $(m > 0)$

so $A_{min} = \dfrac{25\left(\frac{16}{25}\right) + 16}{2\left(\frac{4}{5}\right)}$

$= \dfrac{16 + 16}{\frac{8}{5}} = \boxed{20}$ units^2